Exact Solutions of the Dirac Equation with Coulomb plus a Novel Angle-Dependent Potential

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In this paper the Dirac equation is analytically solved for Coulomb plus a novel angle-dependent potential. The Nikiforov–Uvarov method is used to obtain energy eigenvalues and corresponding eigenfunctions. We also discussed the effect of the angle-dependent part on radial solutions.

Key words: Dirac Equation; Coulomb Potential; Novel Angle-Dependent Potential; Nikiforov–Uvarov Method.
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1. Introduction

The subject of the noncentral potentials has been studied in various fields of nuclear physics and quantum chemistry which may be used to the interactions between deformed pair of nuclei and ring-shaped molecules like benzene [1 – 9]. There has been continues interest in the solutions of Schrödinger, Klein–Gordon, and Dirac equations for some noncentral potentials. These equations are solved by means of different methods for exactly solvable potentials [10 – 35]. Yasuk et al. presented an alternative and simple method for the exact solution of the Klein–Gordon equation in the presence of the noncentral equal scalar and vector potentials by using the Nikiforov–Uvarov method [36]. A spherically harmonic oscillatory ring-shaped potential is proposed and its exactly complete solutions are presented via the Nikiforov–Uvarov method by Zhang et al. [37]. Bayrak et al. [38] and also, Chen et al. [39] presented exact solutions of the Schrödinger equation with the Makarov potential by using the asymptotic iteration method and the partial wave method, respectively. Souza Dutra and Hott solved the Dirac equation by constructing the exact bound state solutions for a mixing of vector and scalar generalized Hartmann potentials [40]. Kandirmaz et al. by using the path integral method investigated the coherent states for a particle in the noncentral Hartmann potential [41]. Chen studied the Dirac equation with the Hartmann potential [42]. The novel angle-dependent (NAD) potential is introduced by Berkdemir [43, 44],

$$V_\theta(\theta) = \frac{\alpha + \beta \sin^2\theta + \eta \sin^4\theta}{\sin^2\theta\cos^2\theta}, \tag{1}$$

therefore the potential is defined here as [40]

$$V(r,\theta) = -\frac{1}{2}\left(\frac{A}{r} - \frac{V_\theta(\theta)}{r^2}\right), \tag{2}$$

where $A = 2Z\alpha$, $\alpha = e^2$ is the fine structure constant in units where $\hbar = c = 1$. In this article, we solve the Dirac equation with this potential by using the Nikiforov–Uvarov method and present the effect of the angle-dependent part on the radial solutions.

2. The Nikiforov–Uvarov Method

To solve second-order differential equations, the Nikiforov–Uvarov method can be used with an appropriate coordinate transformation $s = s(r)$ [45]:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{3}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The following equation is a general form of the Schrödinger-like equation written for any potential [46]:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{[s(1-\alpha_3 s)]^2}\right]\cdot\psi_n(s)=0. \tag{4}$$

According to the Nikiforov–Uvarov method, the eigenfunctions and eigenenergy function become, respectively,

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}\cdot(1-2\alpha_3 s), \tag{5}$$

$$\begin{aligned}&\alpha_2 n-(2n+1)\alpha_5+(2n+1)\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)\\&+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0,\end{aligned} \tag{6}$$

where

$$\begin{aligned}&\alpha_4=\frac{1}{2}(1-\alpha_1), && \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\\&\alpha_6=\alpha_5^2+\xi_1, && \alpha_7=2\alpha_4\alpha_5-\xi_2,\\&\alpha_8=\alpha_4^2+\xi_3, && \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\\&\alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)\\&\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\\&\alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right).\end{aligned} \tag{8}$$

In some problems $\alpha_3=0$. For this type of problems it is

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-\alpha_3)s=L_n^{\alpha_{10}-1}(\alpha_{11}s) \tag{9}$$

and

$$\lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=\mathrm{e}^{\alpha_{13}s}, \tag{10}$$

the solution given in (5) becomes [46]

$$\psi(s)=s^{\alpha_{12}}\mathrm{e}^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{11}$$

3. Dirac Equation with Scalar and Vector Potentials

The time-independent Dirac equation for massive fermions with scalar potential $S(\vec{r})$ and vector potential $V(\vec{r})$ is (with $\hbar=c=1$) [10]

$$[\vec{\alpha}\cdot\vec{p}+\beta(M+S(\vec{r}))]\Psi(\vec{r})=[E-V(\vec{r})]\Psi(\vec{r}), \tag{12}$$

where E is the relativistic energy of the system and $\vec{\alpha}$ and β are the usual 4×4 Dirac matrices

$$\vec{p}=-\mathrm{i}\hbar\vec{\nabla},\qquad \vec{\alpha}=\begin{pmatrix}0&\vec{\sigma}\\\vec{\sigma}&0\end{pmatrix},\qquad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}, \tag{13}$$

where $\vec{\sigma}$ and I are vector Pauli spin matrix and identity matrix, respectively. Using the Pauli–Dirac representation as

$$\Psi(\vec{r})=\begin{pmatrix}\varphi(\vec{r})\\\chi(\vec{r})\end{pmatrix} \tag{14}$$

and substituting (13), (14) into (12), we obtain the following set of coupled equations for the spinor components:

$$\vec{\sigma}\cdot\vec{p}\chi(\vec{r})=[E-V(\vec{r})-M-S(\vec{r})]\varphi(\vec{r}), \tag{15a}$$

$$\vec{\sigma}\cdot\vec{p}\varphi(\vec{r})=[E-V(\vec{r})+M+S(\vec{r})]\chi(\vec{r}). \tag{15b}$$

When the scalar potential $S(\vec{r})$ is equal to the vector potential $V(\vec{r})$ (i.e. $S(r)=V(r)$), (15a) and (15b) become

$$\vec{\sigma}\cdot\vec{p}\chi(\vec{r})=[E-M-2V(\vec{r})]\varphi(\vec{r}), \tag{16a}$$

$$\chi(\vec{r})=\frac{\vec{\sigma}\cdot\vec{p}}{E+M}\varphi(\vec{r}). \tag{16b}$$

Substituting (16b) into (16a), one can obtain the following Schrödinger-like equation:

$$[p^2+2(M+E)V(\vec{r})]\varphi(\vec{r})=[E^2-M^2]\varphi(\vec{r}). \tag{17}$$

By using $\vec{p}=-\mathrm{i}\hbar\vec{\nabla}$ and substituting (2) into above equation, we get

$$\left[-\nabla^2-(M+E)\left(\frac{A}{r}-\frac{V_\theta(\theta)}{r^2}\right)\right]\varphi(r,\theta,\phi)=[E^2-M^2]\varphi(r,\theta,\phi). \tag{18}$$

Now let be

$$\varphi(r,\theta,\phi)=\frac{u(r)}{r}H(\theta)\Phi(\phi). \tag{19}$$

Separating the variables in spherical coordinates, (18) results in

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}+\left\{(E^2-M^2)+\frac{(M+E)A}{r}-\frac{\lambda}{r^2}\right\}\cdot u(r)=0, \tag{20a}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}H(\theta)}{\mathrm{d}\theta}\right)-\left[\frac{m^2}{\sin^2\theta}+(M+E)V_\Theta(\theta)-\lambda\right]H(\theta)=0, \tag{20b}$$

$$\frac{\mathrm{d}^2\Phi(\phi)}{\mathrm{d}\phi^2}+m^2\Phi(\phi)=0, \tag{20c}$$

where m^2 and λ are separation constants. The solution of (20c) is well known as

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\phi}, \qquad m=0,\pm1,\pm2,\ldots \tag{21}$$

4. Solution of Dirac Equation with Coulomb plus NAD Potential

4.1. Solution of the Angle-Dependent Equation

We are now going to derive eigenvalues and eigenfunctions of the polar part of the Dirac equation, i.e. (20b) with the Nikiforov–Uvarov method. Using the transformation $s=\sin^2\theta$, (20b) becomes

$$\frac{\mathrm{d}^2H(s)}{\mathrm{d}s^2}+\frac{2-3s}{2s(1-s)}\frac{\mathrm{d}H(s)}{\mathrm{d}s}+\frac{1}{4s^2(1-s)^2}[m^2(1-s)+(M+E)(\alpha+\beta s+\eta s^2)-\lambda s(1-s)]H(s)=0. \tag{22}$$

By comparing (22) with (4), we get

$$\alpha_1=1,\quad \xi_1=\frac{1}{4}(\lambda+\alpha'),\qquad \alpha_2=\tfrac{3}{2},\quad \xi_2=\frac{1}{4}(m^2-\beta'+\lambda),\qquad \alpha_3=1,\quad \xi_3=\frac{1}{4}(m^2+\eta'), \tag{23}$$

where

$$\alpha'=(M+E)\alpha,\ \ \beta'=(M+E)\beta,\ \ \eta'=(M+E)\eta, \tag{24}$$

and

$$\alpha_4=0,\quad \alpha_5=-\frac{1}{4},\quad \alpha_6=\frac{1}{16}+\xi_1,\quad \alpha_7=-\xi_2,\quad \alpha_8=\xi_3,\quad \alpha_9=\frac{1}{4}(\alpha'+\beta'+\eta'). \tag{25}$$

From (24), (25) and (6), we obtain

$$\lambda=\left(1+2\tilde{n}+\sqrt{m^2+\eta'}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)^2-\left(\sqrt{\frac{1}{4}+\alpha'}\right)^2, \tag{26}$$

where $\tilde{n}$ is a nonnegative integer. For the wave functions of the polar part, from (7) and (8), one obtains

$$\alpha_{10}=1+\sqrt{m^2+\eta'},\quad \alpha_{11}=2+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}+\sqrt{m^2+\eta'},\quad \alpha_{12}=\frac{1}{2}\sqrt{m^2+\eta'},\quad \alpha_{13}=-\frac{1}{4}-\frac{1}{2}\left(\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}+\sqrt{m^2+\eta'}\right), \tag{27}$$

Table 1. Bound state energy eigenvalues using (37) in units of fm^{-1} of the NAD potential with $M=5\,\mathrm{fm}^{-1}$.

$E_{n,\tilde{n},m}$	m	$\tilde{n}$	n
0	0	0	−4.350118
1	0	0	−4.527484
1	0	1	−4.600161
1	1	0	−4.734247
1	1	1	−4.774425
2	0	0	−4.649916
2	0	1	−4.703756
2	1	0	−4.794294
2	1	1	−4.824410
2	2	0	−4.868911
2	2	1	−4.886552
3	0	0	−4.734986
3	0	1	−4.774920
3	1	0	−4.837443
3	1	1	−4.860295

and from (5), we obtain

$$\begin{aligned} H(s) &= s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} P_{\tilde{n}}^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)} \\ &\quad \cdot (1-2\alpha_3 s) \\ &= s^{\frac{1}{2}\sqrt{m^2+\gamma'}}(1-s)^{\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}} \\ &\quad \cdot P_{\tilde{n}}^{\left(\sqrt{m^2+\gamma'},\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)}(1-2s) \end{aligned} \tag{28}$$

or equivalently

$$\begin{aligned} H(\theta) &= C_m(\sin\theta)^{\sqrt{m^2+\gamma'}}(\cos\theta)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}} \\ &\quad \cdot P_{\tilde{n}}^{\left(\sqrt{m^2+\gamma'},\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)}(\cos 2\theta), \end{aligned} \tag{29}$$

where C_m is a normalization constant.

4.2. Solution of the Radial Equation

According to the hydrogen atom, for eigenvalues and the corresponding eigenfunctions of the radial part, i.e. solution of (20a), we have [10]

$$E = M\frac{\left(2n+1+\sqrt{4\lambda+1}\right)^2 - A^2}{\left(2n+1+\sqrt{4\lambda+1}\right)^2 + A^2}, \tag{30}$$

$$u(r) = C_{nl} r^{\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}} \mathrm{e}^{-\sqrt{\varepsilon^2}r} L_n^{2\sqrt{\lambda+\frac{1}{4}}}\left(2\sqrt{\varepsilon^2}r\right), \tag{31}$$

where $\varepsilon^2 = M^2 - E^2$.

For the effect of the angle-dependent part on radial solutions, we substitute (26) into (30) and obtain

$$E_{n,\tilde{n},m} = M\frac{\left(2n+1+\sqrt{4\left(1+2\tilde{n}+\sqrt{m^2+\eta'}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)^2-\left(\sqrt{\frac{1}{4}+\alpha'}\right)^2+1}\right)^2 - A^2}{\left(2n+1+\sqrt{4\left(1+2\tilde{n}+\sqrt{m^2+\eta'}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)^2-\left(\sqrt{\frac{1}{4}+\alpha'}\right)^2+1}\right)^2 + A^2}. \tag{32}$$

When $\alpha = \beta = \eta = 0$, from (26) we have $\lambda = l(l+1)$, where $l = 1+2\tilde{n}+|m|$ and the Coulomb plus a NDA potential reduces to the Coulomb potential, and the energy eigenvalues are obtained as follows [30, 41]:

$$E_{(\text{Coulomb})} = M\frac{n'^2 - Z^2\alpha^2}{n'^2 + Z^2\alpha^2}, \tag{33}$$

where $n' = n+l+1 = 1,2,3,\ldots$. Finally, we can write $\varphi(\vec{r})$ as

$$\begin{aligned} \varphi(\vec{r}) &= \frac{u(r)}{r}H(\theta)\Phi(\phi) \\ &= \frac{C_{nm}}{\sqrt{2\pi}} r^{\sqrt{\lambda+\frac{1}{4}}-\frac{1}{2}} \mathrm{e}^{-\sqrt{\varepsilon^2}r} L_n^{2\sqrt{\lambda+\frac{1}{4}}}\left(2\sqrt{\varepsilon^2}r\right) \\ &\quad \times (\sin\theta)^{\sqrt{m^2+\gamma'}}(\cos\theta)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}} \\ &\quad \cdot P_{\tilde{n}}^{(\sqrt{m^2+\gamma'},\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'})}(\cos 2\theta)\mathrm{e}^{\mathrm{i}m\phi}, \end{aligned} \tag{34}$$

where C_{nm} is the normalization constant. From (16b) and (14), the spinor wave function becomes

$$\begin{aligned} \Psi(\vec{r}) &= \begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \frac{C_{nm}}{\sqrt{2\pi}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix} \\ &\quad \cdot r^{\sqrt{\lambda+\frac{1}{4}}-\frac{1}{2}} \mathrm{e}^{-\sqrt{\varepsilon^2}r} L_n^{2\sqrt{\lambda+\frac{1}{4}}}\left(2\sqrt{\varepsilon^2}r\right) \\ &\quad \times (\sin\theta)^{\sqrt{m^2+\gamma'}}(\cos\theta)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}} \\ &\quad \cdot P_{\tilde{n}}^{\left(\sqrt{m^2+\gamma'},\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)}(\cos 2\theta)\mathrm{e}^{\mathrm{i}m\phi}. \end{aligned} \tag{35}$$

At the end, when $S(\vec{r}) = -V(\vec{r})$, one can find the lower spinor component of the Dirac equation. To avoid repetition, we use below transformations in (16a) and (16b) as in [41]:

$$\begin{aligned} &\phi(r) \to \chi(r), \\ &\chi(r) \to -\phi(r), \\ &V(r) \to -V(r), \\ &E \to -E. \end{aligned} \tag{36}$$

By using (32), the relativistic energy spectrum related to the lower spinor component is obtained as

$$E_{n,\tilde{n},m} = M\frac{A^2 - \left(2n+1+\sqrt{4\left(1+2\tilde{n}+\sqrt{m^2+\eta'}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)^2-\left(\sqrt{\frac{1}{4}+\alpha'}\right)^2+1}\right)^2}{A^2 + \left(2n+1+\sqrt{4\left(1+2\tilde{n}+\sqrt{m^2+\eta'}+\sqrt{\frac{1}{4}+\alpha'+\beta'+\eta'}\right)^2-\left(\sqrt{\frac{1}{4}+\alpha'}\right)^2+1}\right)^2}. \quad (37)$$

Some numerical results of (37) are presented in Table 1. The parameters are taken as $M = 5\text{fm}^{-1}$, $Z = 1$, and $\alpha = \beta = \eta = 1$.

5. Conclusion

We have studied the exact solutions of the Dirac equation with the Coulomb plus a novel angle-dependent potential using the Nikiforov–Uvarov method when the scalar potential and the vector potential are equal. The bound states energy eigenvalues and the corresponding wave functions are obtained. We also showed that the results can be used to evaluate the binding energies of the noncentral potential for diatomic molecules in the relativistic framework and when $\alpha = \beta = \eta = 0$, they are in good agreement with the results of [30, 41].

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[1] H. Hartmann, Theor. Chim. Acta **24**, 201 (1972).
[2] H. Hartmann, R. Schuck, and J. Radtke, Theor. Chim. Acta **46**, 1 (1976).
[3] A. A. Makarov, J. A. Smorodinsky, K. H. Valiev, and P. Winternitz, Nuovo Cimento A **52**, 1061 (1967).
[4] A. Khare and R. K. Bhaduri, Am. J. Phys. **62**, 1008 (1994).
[5] X. A. Zhang, K. Chen, and Z. L. Duan, Chin. Phys. **14**, 42 (2005).
[6] S. H. Dong, G. H. Sun, and M. Lozada-Cassou, Phys. Lett. A **328**, 299 (2005).
[7] C. Y. Chen and S. H. Dong, Phys. Lett. A **335**, 374 (2005).
[8] A. D. Alhaidari, J. Phys. A: Math. Gen. **38**, 3409 (2005).
[9] M. Kibler, L. G. Mardoyan, and G. S. Pogosyan, Int. J. Quantum Chem. **52**, 1301 (1994).
[10] W. Gereiner, Relativistic Quantum Mechanics, Wave Equations, third edition, Springer, Berlin, Heidelberg 2000.
[11] C. Quense, Int. J. Mod. Phys. A **6**, 1567 (1991).
[12] B. Sutherland, Phys. Rev. Lett. **80**, 3678 (1998).
[13] C. F. Hou, Y. Li, and Z. X. Zhou, Acta Phys. Sinica **48**, 1999 (1999) (in Chinese).
[14] J. Y. Guo, Acta Phys. Sin. **51**, 1453 (2002) (in Chinese).
[15] M. C. Zhang and Z. B. Wang, Chin. Phys. Lett. **22**, 2994 (2005).
[16] F. Yaşuk, C. Berkdemir, and A. Berkdemir, J. Phys. A: Math. Gen. **38**, 6579 (2005).
[17] C. S. Jia, P. Guo, and X. L. Peng, J. Phys. A: Math. Gen. **39**, 7737 (2006).
[18] W. C. Qiang, R. S. Zhou, and Y. Gao, J. Phys. A: Math. Theor. **40**, 1677 (2007).
[19] A. Arda, R. Sever, and C. Tezcan, Chinese J. Phys. **48**, 27 (2010).
[20] A. Soylu, O. Bayrak, and I. Boztosun, J. Math. Phys. **48**, 082302 (2007).
[21] S. H. Dong and X. Y. Gu, J. Phys.: Conf. Ser. **96**, 012109 (2008).
[22] G. F. Wei and S. H. Dong, Phys. Lett. A **373**, 49 (2008).
[23] A. Soylu, O. Bayrak, and I. Boztosun, J. Phys. A: Math. Theor. **41**, 065308 (2008).
[24] Y. Xu, S. He, and C. S. Jia, J. Phys. A: Math. Theor. **42**, 198002 (2009).
[25] M. Hamzavi, H. Hassanabadi, and A. A. Rajabi, Int. J. Mod. Phys. E **19**, 2189 (2010).
[26] X. Y. Liu, G. F. Wei, and C. Y. Long, Int. J. Theor. Phys. **48**, 463 (2009).
[27] T. Chen, Y. F. Diao, and C. S. Jia, Phys. Scr. **79**, 065014 (2009).
[28] O. Aydoğdu and R. Sever, Phys. Scr. **80**, 015001 (2009).
[29] S. M. Ikhdair, J. Math. Phys. **51**, 023525 (2010).
[30] M. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Few-Body Syst. **48**, 171 (2010).
[31] C. Berkdemir and R. Sever, J. Phys. A: Math. Theor. **41**, 045302 (2008).
[32] M. Hamzavi, H. Hassanabadi, and A. A. Rajabi, Mod. Phys. Lett. A **25**, 2447 (2010).
[33] N. Kandirmaz and R. Sever, Phys. Scr. **81**, 035302 (2010).

[34] H. Xian-Quan, L. Guang, W. Zhi-Min, N. Lian-Bin, and M. Yan, Commun. Theor. Phys. (Beijing, China) **53**, 242 (2010).
[35] G. H. Sun and S. H. Dong, Mod. Phys. Lett A **26**, 351 (2011).
[36] F. Yasuk, A. Durmus, and I. Boztosun, J. Math. Phys. **47**, 082302 (2006).
[37] M. C. Zhang, G. H. Sun, and S. H. Dong, Phys. Lett. A **374**, 704 (2010).
[38] O. Bayrak, M. Karakoc, I. Boztosun, and R. Sever, Int. J. Theor. Phys. **47**, 3005 (2008).
[39] C. Y. Chen, C. L. Liu, and F. L. Lu, Phys. Lett. A **374**, 1346 (2010).
[40] A. de Souza Dutra and M. Hott, Phys. Lett. A **356**, 215 (2006).
[41] N. Kandirmaz and N. Ünal, Theor. Math. Phys, **155**, 884 (2008).
[42] C. Y. Chen, Phys. Lett. A **339**, 283 (2005).
[43] C. Berkdemir, J. Math. Chem. **46**, 139 (2009).
[44] C. Berkdemir and Y. F. Cheng, Phys. Scr. **79**, 035003 (2009).
[45] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physic, Birkhäuser, Berlin 1988.
[46] C. Tezcan and R. Sever, Int. J. Theor. Phys. **48**, 337 (2009).